Optimal strong Mal'cev conditions for congruence meet-semidistributivity in locally finite varieties

Jelena Jovanović, Petar Marković, Ralph McKenzie, and Matthew Moore

Abstract. Locally finite, congruence meet-semidistributive varieties have been characterized by numerous Mal'cev conditions and, recently, by two strong Mal'cev conditions. We provide three new strong Mal'cev characterizations and a new Mal'cev characterization each of which improves the known ones in some way.

1. Introduction

The various conditions which are equivalent to congruence meet-semidistributivity in locally finite varieties of algebras have been explored in several previous papers and books [5], [7], [11], [15], [20], [13]. The reason for this activity is that congruence meet-semidistributive varieties are a very general, and yet well behaved class of varieties. For instance, this condition is equivalent to congruence neutrality [11] and [15]; in locally finite varieties it is characterized by omitting tame congruence theory types **1** and **2** [7]; the Park's conjecture is true in congruence meet-semidistributive varieties [20], see also [12]; it characterizes the algebraic duals of the finite relational structures $\mathbb{A}$ such that the constraint satisfaction problem with template $\mathbb{A}$ can be accurately solved by using only the local consistency checking [14], [2], see also [1].

We are concerned in this paper with an optimal strong Mal'cev characterization for congruence meet-semidistributivity. Siggers proved in [19] that the weaker property, having a Taylor term (characterized in locally finite varieties by omitting type **1**) is a strong Mal'cev property, when restricted to locally finite varieties. The Siggers' result was a big surprise at the time of publication and spurred an investigation of what other properties which were hitherto known to have a Mal'cev characterization would have a strong Mal'cev characterization in locally finite varieties. Congruence meet-semidistributivity was proved to be a strong Mal'cev property in the case of locally finite varieties in [13], while many other natural properties were proved not to have a strong Mal'cev characterization in the same paper. The paper [10] settled the question of optimal (syntactically simplest) strong Mal'cev characterization of having a

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Taylor term (= omitting type **1**) in locally finite varieties. The paper [8] managed to prove that there are no strong Mal'cev characterizations of congruence meet-semidistributive locally finite varieties in the language with at most one ternary operation and all other operations with arities less than 3. In the case of conditions with two ternary operations, the paper [8] isolates one candidate condition and proves that every strong Mal'cev characterization with two ternary operations of congruence meet-semidistributivity in locally finite varieties must imply this one. In the present paper we prove that that condition isolated by [8] indeed does characterize congruence meet-semidistributivity in locally finite varieties. We also find a strong Mal'cev characterization in the language of one operation of arity 4. The two strong Mal'cev conditions we find are, thus, syntactically optimal.

Our paper is organized as follows: In Section 2, we give a list of definitions which will be used, beyond the classical universal algebra definitions and results which we assume the reader to be familiar with. Those readers who are not familiar with them are advised to check out the textbooks [4], [18] and/or [3]. To follow all proofs of cited results, the reader needs some knowledge of tame congruence theory developed in [7] and commutator theory in [6], but no knowledge of either is needed to follow our arguments if the reader is content to trust the cited theorems. We continue Section 2 with definitions and a review of the constraint satisfaction problem terminology and results, particularly the main result of [1], which will be the main tool used to prove the harder direction of our two Mal'cev characterizations. In the same section we recall a result of commutator theory from [9] which we will use to prove the easier direction (that the Mal'cev condition implies congruence meet-semidistributivity) in our results. We conclude Section 2 with definitions and review of relevant results about Mal'cev conditions which will be used in the paper.

In Section 3 we prove the main theorem of this paper. It gives a strong Mal'cev characterization of locally finite varieties in the language which consists of one operation of arity 4. This is an optimal strong Mal'cev characterization of congruence meet-semidistributive locally finite varieties, as per terminology of [10]. The other optimal language for such a strong Mal'cev characterization, namely the language which consists of two ternary operation symbols is also realized, which is a corollary of the main theorem.

We conclude the paper with a list of topics for further research in Section 4, speculating on directions in which our results could be further improved. Several partial results in those directions are proved.

2. Definitions and background

We begin by defining the property this paper chiefly investigates and attempts to characterize.

Definition 2.1. An algebra is congruence meet-semidistributive if for any congruences $\alpha, \beta, \gamma \in \text{Con } \mathbf{A}$, the following implication holds:

$$\alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \alpha \wedge \beta = \alpha \wedge (\beta \vee \gamma).$$

A variety $\mathcal{V}$ is congruence meet-semidistributive if every algebra in $\mathcal{V}$ is congruence meet-semidistributive.

We turn to definitions of the constraint satisfaction problem and a $(2,3)$-minimal instance of it. We follow [1] as we will use the main result of that paper a lot. The definition we give below is Definition 3.1 of [1]:

Definition 2.2. An instance of the constraint satisfaction problem (CSP) is a triple $(V; A; \mathcal{C})$ with

- V a nonempty, finite set of variables,
- A a nonempty, finite domain,
- $\mathcal{C}$ a finite nonempty set of constraints, where each constraint is a subset C of A^W. Here W is a subset of V called the scope of C and the cardinality $|W|$ of W is referred to as the arity of C.

An instance is *trivial* if it contains the empty constraint. The instance $(V; A; \mathcal{C})$ has a solution, that is, a function $f : V \to A$ such that, for each constraint $C \in \mathcal{C}$, such that the scope of C is $W \subseteq V$, the restriction $f \restriction_W$ is in C. Next we define a 2-consistent and a $(2,3)$-minimal instance:

Definition 2.3. An instance of CSP $(V; A; \mathcal{C})$ is 2-consistent, if for every $U \subseteq V$ such that $|U| \leq 2$ and every pair of constraints $C, D \in \mathcal{C}$ such that U is contained in the scopes of both C and D, the restrictions $C \restriction_U = D \restriction_U$. An instance of CSP $(V; A; \mathcal{C})$ is $(2,3)$-minimal if it is 2-consistent and every at most 3-element subset of V is contained in the scope of some constraint in $\mathcal{C}$.

In order to make our proofs easier to read we introduce a convention that the elements of A^W (mappings from W to A) are written as vector columns. This allows us to see better how to apply an operation which acts coordinatewise to several such vectors. When we describe the constraint $C \subseteq A^W$, we linearly order the elements of $W = \{x_{i_1}, \dots, x_{i_k}\}$. Then we write $\rho_{i_1,\dots,i_k} = R$, for some previously fixed $R \subseteq A^{|W|}$, which means that the uppermost coordinate of the vector column in R is the image of x_{i_1}, below it the image of x_{i_1}, and so on. In some cases, to save space we will use the transpose of the vector column, which will be denoted by $[a_1, \dots, a_k]^T$.

Definition 2.4. Let $\mathbb{A} = \langle A; \Gamma \rangle$ be a relational structure. An instance of the constraint satisfaction problem $CSP(\mathbb{A})$ is any instance of the CSP $(V; D; \mathcal{C})$ such that for each constraint $C \in \mathcal{C}$, the relation C is equal to a permutation of coordinates $C' \in \Gamma$. The structure $\mathbb{A}$ is called the *template* of $CSP(\mathbb{A})$.

We silently assume that all Γ contain the equality relation (to allow using the relations obtained by identification of variables).

Let $\mathbf{A}$ be an algebra. When $\Gamma \subseteq \mathsf{SP}_{fin}(\mathbf{A})$, then we say that $CSP(\langle A; \Gamma\rangle)$ is compatible with $\mathbf{A}$. The following result (Corollary 6.5 of [1]) is the one that has as a consequence the main theorem of [1]:

Theorem 2.5. *Let $\mathbf{A}$ be an idempotent finite algebra which generates a congruence meet-semidistributive variety. Then for every $CSP(\langle A; \Gamma\rangle)$ which is compatible with $\mathbf{A}$, every $(2,3)$-minimal instance of $CSP(\langle A; \Gamma\rangle)$ which is not trivial has a solution.*

By a *strong Mal'cev condition* we mean a finite set of identities in some language. Informally, a strong Mal'cev condition is *realized* in an algebra $\mathbf{A}$ (or variety $\mathcal{V}$) if there is a way to interpret the function symbols appearing in the condition as term operations of $\mathbf{A}$ (or $\mathcal{V}$) so that the identities in the Mal'cev condition become true equations in $\mathbf{A}$ (or $\mathcal{V}$). A Mal'cev condition is a sequence $\{C_n : n \in \omega\}$ of strong Mal'cev conditions such that any variety which realizes C_n must also realize C_{n+1} for all $n \in \omega$. We say that the variety $\mathcal{V}$ realizes the Mal'cev condition $\{C_n : n \in \omega\}$ if there exists an $n \in \omega$ such that $\mathcal{V}$ realizes C_n.

We recall the following characterization of congruence meet-semidistributivity, proved in [9], Theorem 8.1., (1) $\Leftrightarrow$ (10).

Theorem 2.6. *Let $\mathcal{V}$ be a variety. $\mathcal{V}$ is congruence meet-semidistributive iff $\mathcal{V}$ satisfies an idempotent Mal'cev condition which fails in any variety of modules.*

We give two Mal'cev conditions and two strong Mal'cev conditions which will be of further use to us:

We say that a variety has Jónsson terms if there exists $n \geq 2$ such that $\mathcal{V}$ realizes the strong condition $CD(n)$. $CD(n)$ is in the language $\{d_0, d_1, \ldots, d_n\}$ consisting of ternary symbols, and consists of identities

$$\begin{aligned}
d_0(x,y,z) &\approx x, & \\
d_i(x,y,x) &\approx x & &\text{for all } 0 \leq i \leq n,\\
d_i(x,y,y) &\approx d_{i+1}(x,y,y) & &\text{for all even } i \text{ such that } 0 \leq i \leq n,\\
d_i(x,x,y) &\approx d_{i+1}(x,x,y) & &\text{for all odd } i \text{ such that } 0 \leq i \leq n,\\
d_n(x,y,z) &\approx z. &
\end{aligned}$$

We say that a variety has a weak near-unanimity term if there exists $n \geq 3$ such that $\mathcal{V}$ realizes the strong condition $WNU(n)$. $WNU(n)$ is in the language $\{w\}$, the arity is $ar(w) = n$, and consists of identities

$$w(x,x,\ldots,x) \approx x,$$
$$w(y,x,x,\ldots,x) \approx w(x,y,x,\ldots,x) \approx \ldots \approx w(x,x,\ldots,x,y).$$

It was proved in the 1960s by Jónsson that a variety is congruence distributive iff it realizes $CD(n)$ for some $n \geq 2$. This kind of equivalence is usually called a Mal'cev characterization of some property. In [17] it was proved that any locally finite variety $\mathcal{V}$ is congruence meet-semidistributive iff

$\mathcal{V}$ realizes the strong Mal'cev conditions $WNU(n)$ for all but finitely many $n \in \omega \setminus \{0,1,2\}$. This was a Mal'cev characterization of congruence meet-semidistributivity *within the class of locally finite varieties.*

Various Mal'cev conditions may be equivalent when restricted to locally finite varieties, but inequivalent in general. We can measure the syntactic strength of these conditions, namely their position in the lattice of interpretability types. This is a preorder which is induced by the relation "condition Σ_1 is realized in the variety of all models of condition Σ_2". An equivalent perspective at this preorder is proved in the following proposition.

Proposition 2.7. *Let Σ_1 and Σ_2 be strong Mal'cev conditions. The following statements are equivalent:*

- *Any variety which realizes Σ_1 must also realize Σ_2.*
- *Σ_1 is realized in the variety $\mathcal{V} = \mathrm{Mod}(\Sigma_2)$.*

Proof. One direction follows since the variety $\mathcal{V}$ (which treats Σ_2 as its base of equations) realizes Σ_2, while the other direction follows since the composition of the realization of condition Σ_1 into $\mathrm{Mod}(\Sigma_2)$ with the realization of Σ_2 in some $\mathcal{V}$ is also a realization of Σ_1. □

We will write $\Sigma_2 \preceq \Sigma_1$ in the case when either condition of the above Proposition is satisfied and say that the strong Mal'cev conditition Σ_1 is stronger than the condition Σ_2. When $\Sigma_1 \preceq \Sigma_2$ and $\Sigma_2 \preceq \Sigma_1$, we say that those two strong Mal'cev conditions are equivalent and write $\Sigma_2 \sim \Sigma_1$.

However, when one restricts the scope of varieties one cares about, one gets a lot of identification. We write $\Sigma_2 \preceq_{lf} \Sigma_1$, meaning that "any locally finite variety which realizes Σ_1 must also realize Σ_2". The equivalence induced by the preorder $\preceq_{lf}$ is denoted by $\sim_{lf}$. Often $\sim$-inequivalent conditions become equivalent with respect to $\sim_{lf}$.

Now we state the strong Mal'cev characterization of congruence meet-semidistributivity in locally finite varieties which was proved in [13]:

Theorem 2.8 (Theorem 2.8 of [13])**.** *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff $\mathcal{V}$ realizes the strong Mal'cev condition given by:*

$$\begin{aligned} &p(x,x,x) \approx x \approx w(x,x,x,x),\\ &p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx w(x,x,x,y)\\ &\approx w(x,x,y,x) \approx w(x,y,x,x) \approx w(y,x,x,x). \end{aligned} \tag{SM 1}$$

Here is an easier way to remember the above condition: p is a ternary weak near-unanimity, w is a 4-ary weak near-unanimity and $t(x,x,y) \approx w(x,x,x,y)$.

Another strong Mal'cev characterization of congruence meet-semidistributivity, due to Janko and Maróti, is a direct corollary of Theorem 2.8.

Corollary 2.9. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff $\mathcal{V}$ realizes the strong Mal'cev condition given by:*

$$\begin{aligned} &p(x,x,x) \approx q(x,x,x) \approx r(x,x,x) \approx x, \\ &p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,x,y) \\ &\approx q(x,y,x) \approx r(x,y,x) \approx r(y,x,x), \\ &q(x,y,y) \approx r(x,x,y) \end{aligned} \qquad \text{(SM 2)}$$

Proof. On the one hand, if $\mathcal{V}$ is congruence meet-semidistributive, then from Theorem 2.8, by taking $q(x,y,z) = w(x,x,y,z)$ and $r(x,y,z) = w(x,y,z,z)$, we get that (SM 2) holds in $\mathcal{V}$.

On the other hand, if the condition (SM 2) fails in any nontrivial $\mathbf{R}$-module $\mathbf{M}$, as can be seen by setting $p(x,y,z) = \alpha_1 x + \alpha_2 y + \alpha_3 z$, $q(x,y,z) = \beta_1 x + \beta_2 y + \beta_3 z$ and $r(x,y,z) = \gamma_1 x + \gamma_2 y + \gamma_3 z$. Now the second string of equations of (SM 2), by evaluating $x = 0$ we get that $\alpha_1 = \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_1 = \gamma_2 =: \alpha$. The final equation implies that $\beta_1 = 2\alpha$, while the first equation (idempotence) now implies that $3\alpha = 4\alpha = 1$, so $\alpha = 0$ and $3\alpha = 1$, which implies that $0 = 1$, and so for any element $x \in M$, $x = 1x = 0x = 0$, so $\mathbf{M}$ is trivial. □

3. Optimal strong Mal'cev characterizations of congruence meet-semidistributivity

Let all us define a sequence of positive integers $\{w_n : n \geq 1\}$ recursively, by

$$w_1 = 4 \quad \text{and} \quad w_{n+1} = 3(n+1)(2^{w_n} - 1) + 1$$

for all $n > 0$. We prove the following Ramsey-style lemma:

Lemma 3.1. *If $P(w_n) \setminus \{\emptyset\}$ is colored by φ in n colors (i. e. $\varphi : (P(w_n) \setminus \{\emptyset\}) \to \{1, 2, \ldots, n\}$), then there exist distinct subsets $A_1, \ldots, A_7 \in P(w_n) \setminus \{\emptyset\}$ such that*

- *$A_1 \cap A_i = \emptyset$ for all $i > 1$;*
- *$A_i \cap A_j = \emptyset$ whenever $2 \leq i \leq 4$ and $j \in \{2, 3, 4, i+3\} \setminus \{i\}$;*
- *$A_2 \subseteq A_6 \cap A_7$, $A_3 \subseteq A_5 \cap A_7$ and $A_4 \subseteq A_5 \cap A_6$;*
- *none of A_5, A_6, A_7 is contained in any other among the seven sets and*
- *$\varphi(A_1) = \varphi(A_2) = \varphi(A_3) = \varphi(A_4) = \varphi(A_5) = \varphi(A_6) = \varphi(A_7)$.*

Proof. The proof is by an induction on n. If $n = 1$, then $w_1 = 4$ and we take $A_1 = \{0\}$, $A_2 = \{1\}$, $A_3 = \{2\}$, $A_4 = \{3\}$, $A_5 = \{2, 3\}$, $A_6 = \{1, 3\}$ and $A_7 = \{1, 2\}$.

Assume that $n > 1$ and that the claim is true for $n - 1$. Since $w_n = 3n \cdot (2^{w_{n-1}} - 1) + 1$, without loss on generality we may assume that at least $3 \cdot 2^{w_{n-1} - 1} + 1$ singleton subsets of w_n are colored with the color n. Select $A \subseteq w_n$ such that $|A| = 3(2^{w_{n-1}} - 1)$, all singleton subsets of A are colored with n, and select $a \in w_n \setminus A$ such that $\varphi(\{a\}) = n$.

First we will define the order embedding ψ_1 of the poset $P(w_{n-1})$ into the poset $P(A)$ which preserves incomparability and disjointness. Thus any family of seven subsets of $P(w_{n-1})$ which satisfies the first four conditions in the statement of the lemma maps by ψ_1 into such a family of subsets of A.

For any nonempty set $X \in P(w_{n-1})$, select $\tau(X) \in P(A)$ such that $|\tau(X)| = 3$, and such that if $X \neq Y$ and $X, Y \in P(w_{n-1})$ are nonempty, then $\tau(X) \cap \tau(Y) = \emptyset$. Moreover, let $\tau(\emptyset) = \emptyset$. Now define $\psi_1(X) = \bigcup\{\tau(Y) : Y \subseteq X\}$. Distinct subsets of w_{n-1} have distinct powersets, so this is an injective map. If $X \subseteq Y \subseteq w_{n-1}$ then $P(X) \subseteq P(Y)$, so ψ_1 is order-preserving. If X and Y are incomparable, then $X \setminus Y$ is a nonempty set, as is $Y \setminus X$, and $\emptyset \neq \tau(X \setminus Y) \subseteq \psi_1(X) \setminus \psi_1(Y)$, while $\emptyset \neq \tau(Y \setminus X) \subseteq \psi_1(Y) \setminus \psi_1(X)$. Finally, we show that X and Y are disjoint iff $\psi_1(X)$ and $\psi_1(X)$ are. Let $X, Y \subseteq w_{n-1}$ be disjoint. Then $P(X) \cap P(Y) = \{\emptyset\}$, so $\psi_1(X)$ and $\psi_1(Y)$ intersect in $\tau(\emptyset) = \emptyset$, i.e. $\psi_1(X) \cap \psi_1(Y) = \emptyset$. On the other hand, if X and Y are not disjoint, then $\emptyset \neq \tau(X \cap Y) \subseteq \psi_1(X) \cap \psi_1(Y)$. Also, note that $|A| = 3(2^{w_{n-1}} - 1)$, so it is just big enough in size to enable the selection of all $\tau(X)$ which are mutually disjoint three-element sets, and thus $\psi_1(w_{n-1}) = A$.

Let $\tau(X) = \{b, c, d\}$ for some $\emptyset \subsetneq X \subseteq w_{n-1}$. If $\varphi(\psi_1(X) \setminus \{b\}) = \varphi(\psi_1(X) \setminus \{c\}) = \varphi(\psi_1(X) \setminus \{d\}) = n$, then $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$, $A_4 = \{d\}$, $A_5 = \psi_1(X) \setminus \{b\}$, $A_6 = \psi_1(X) \setminus \{c\}$ and $A_7 = \psi_1(X) \setminus \{d\}$ satisfy the conclusion of the lemma. So we may assume that for every $\emptyset \subsetneq X \subseteq w_{n-1}$, there exists $x \in \tau(X)$ such that $\varphi(\psi_1(X) \setminus \{x\}) \neq n$.

Select $\psi : P(w_{n-1}) \to P(A)$ such that $\psi(\emptyset) = \emptyset$, while for all X such that $\emptyset \subsetneq X \subseteq w_{n-1}$, $\psi(X) = \psi_1(X) \setminus \{x\}$, where $x \in \tau(X)$ is such that $\varphi(\psi(X)) \neq n$. We know that $X \subseteq Y$ in $P(w_{n-1})$ implies $X = Y$ or $\psi(X) \subseteq \psi_1(X) \subseteq \bigcup\{\psi_1(Z) : Z \subsetneq Y\} \subsetneq \psi(Y)$, so we get that ψ is an order-preserving map between $P(w_{n-1})$ and $\psi(P(w_{n-1}))$. If $X \not\subseteq Y$, then $X \setminus Y \neq \emptyset$, so $\tau(X \setminus Y) \subseteq \psi_1(X) \setminus \psi_1(Y)$ and thus $|\psi_1(X) \setminus \psi_1(Y)| \geq 3$. Therefore $|\psi(X) \setminus \psi(Y)| \geq |\psi(X) \setminus \psi_1(Y)| \geq |\psi_1(X) \setminus \psi_1(Y)| - 1 \geq 2$, and ψ preserves the non-containment, so it is an order-isomorphism between $P(w_{n-1})$ and $\psi(P(w_{n-1}))$. Since $\psi(X) \cap \psi(Y) \subseteq \psi_1(X) \cap \psi_1(Y)$, thus $X \cap Y = \emptyset$ implies $\psi_1(X) \cap \psi_1(Y) = \emptyset$ which in turn implies $\psi(X) \cap \psi(Y) = \emptyset$. On the other hand, if $X \cap Y \neq \emptyset$, then $|\psi_1(X) \cap \psi_1(Y)| \geq 3$, so $|\psi(X) \cap \psi(Y)| \geq |\psi_1(X) - \psi_1(Y)| - 2 > 0$. Thus $\psi(X)$ and $\psi(Y)$ are disjoint iff X and Y are.

Now we define the coloring φ_1 of $P(w_{n-1}) \setminus \{\emptyset\}$ as $\varphi_1(X) = \varphi(\psi(X))$ whenever $\emptyset \neq X \subseteq w_{n-1}$. By the inductive assumption, there exist subsets $B_1, \dots, B_7 \subseteq w_{n-1}$ such that they satisfy the conclusion of the lemma with respect to the coloring φ_1. We note that all B_i must be nonempty subsets of w_{n-1} by the first three conditions. Thus $A_i := \psi(B_i)$, for $1 \leq i \leq 7$, satisfy the conclusion of the lemma with respect to φ, the first four conditions since we proved that ψ is an order-isomorphism between $P(w_{n-1})$ and $\psi(P(w_{n-1}))$ such that ψ and ψ^{-1} preserve disjointness. $\square$

Now we apply Lemma 3.1 to prove a strong Mal'cev characterization of congruence meet-semidistributivity:

Theorem 3.2. *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ is locally finite and congruence meet-semidistributive, then $\mathcal{V}$ realizes the strong Mal'cev condition* (SM 3) *given by*

$$\begin{aligned}
&t(x,x,x,x) \approx x \\
&t(y,x,x,x) \approx t(x,y,x,x) \approx t(x,x,y,x) \approx \\
&t(x,x,x,y) \approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x)
\end{aligned} \tag{SM 3}$$

On the other hand, if $\mathcal{V}$ realizes the strong Mal'cev condition (SM 3)*, then $\mathcal{V}$ is congruence meet-semidistributive.*

Proof. Let $\mathbf{M}$ be a left $\mathbf{R}$-module which satisfies (SM 3). Then $t^{\mathbf{M}}(x,y,z,u) = \alpha x + \beta y + \gamma z + \delta u$, and the evaluation $x = 0, y = y$ implies that $\alpha = \beta = \gamma = \delta = \alpha + \beta$, and hence $\alpha = 0$. But idempotence implies that $x = t^{\mathbf{M}}(x,x,x,x) = 4\alpha x = 0x = 0$, so $\mathbf{M}$ is trivial. Thus $\mathcal{V}$ is congruence meet-semidistributive, according to Theorem 2.6, and the second statement is proved.

To prove the first statement, let $\mathcal{V}$ be a localy finite congruence meet-semidistributive variety. Let $\mathcal{W}$ be the idempotent reduct of $\mathcal{V}$, which is the variety whose clone is the clone of idempotent term operations of $\mathcal{V}$ and whose fundamental operations are the distinct elements of this clone. Since congruence meet-semidistributivity can be characterized by an idempotent Mal'cev condition, $\mathcal{W}$ is a locally finite, idempotent, congruence meet-semidistributive variety. From idempotence follows that there exist term operations p and q in the language of $\mathcal{V}$ which satisfy identities (SM 3) in $\mathcal{V}$ iff there exist such term operations which satisfy (SM 3), except for idempotence, in $\mathcal{W}$.

Let $\mathbf{F}$ be the two-generated free algebra in $\mathcal{W}$, freely generated by x and y. Let $|F| = n$. We define some subalgebras of $\mathbf{F}^2$ (i.e. compatible binary relations with the operations of $\mathbf{F}$):

$$\begin{aligned}
E &= \mathrm{Sg}^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right) \\
\leq &= \mathrm{Sg}^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right) \\
G &= \mathrm{Sg}^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right)
\end{aligned}$$

We claim that the relation G is actually equal to the full product $F \times F$. To prove this, let $r(x,y), s(x,y) \in F$ be arbitrary. Then

$$\begin{aligned}
s^{\mathbf{F}^2}\left(r^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right), r^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right)\right) = \\
s^{\mathbf{F}^2}\left(\begin{bmatrix} r(x,y) \\ x \end{bmatrix}, \begin{bmatrix} r(x,y) \\ y \end{bmatrix}\right) = \begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}
\end{aligned}$$

Now we define eleven more ternary compatible relations of $\mathbf{F}$:

$$R_1 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}\right)$$

$$R_2 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}\right)$$

$$R_3 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \end{bmatrix}\right)$$

$$R_4 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \\ y \end{bmatrix}\right)$$

$$R_5 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \\ y \end{bmatrix}\right)$$

$$R_6 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \\ y \end{bmatrix}\right)$$

$$R_7 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \end{bmatrix}\right)$$

$$R_8 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \end{bmatrix}\right),$$

while the final three relations are defined by $R_9 := \{[p,q,r]^T : [p,q]^T \in E\}$, $R_{10} := \{[p,q,r]^T : [p,q]^T \in \leq\}$ and $R_{11} = F \times F \times F$.

We note the following facts: The projection of R_1 to any pair of coordinates is E. The projection of R_2 to the first two coordinates is E, while the other two projections of R_2 are equal to $\leq$. The projection of R_3 to the first two coordinates is $\leq$, while the other two projections of R_3 are equal to E. The projection of R_4 to any pair of coordinates is $\leq$. The projection of R_5 to the last two coordinates is G, while the other two projections of R_5 are equal to $\leq$. The projection of R_6 to the first two coordinates is G, while the other two projections of R_6 are equal to $\leq$. The projection of R_7 to the first two coordinates is $\leq$, the projection of R_7 to the first and the last coordinate is E, while the projection of R_7 to the last two coordinates is G. The projection of R_8 to the last two coordinates is G, while the projection of R_8 to any other pair of coordinates is E. Finally, all projections of R_9, R_{10} and R_{11} to a pair of coordinates are equal to $G = F \times F$, except for the projection of R_9 to the first two coordinates, which is equal to E and the projection of R_{10} to the first two coordinates, which is equal to $\leq$. The last statement follows from the subdirectness of the binary relations E and $\leq$.

We describe one more relation, the subpower U of arity 7:

$$U = \mathrm{Sg}^{\mathbf{F}^7}\left(\begin{bmatrix} x \\ x \\ x \\ y \\ y \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \\ x \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \\ x \\ x \\ x \end{bmatrix}\right)$$

Note that the projection of U to the first and any other coordinate, to any pair among the second, third and fourth coordinate, and also to ith and $(i+3)$rd coordinate for any $2 \leq i \leq 4$, equals E. The projections of U to any pair among the last three coordinates equals G, while its projection to any other pair of coordinates equals $\leq$.

We describe an instance $(V, F, \mathcal{C})$ of the constraint satisfaction problem with the template $\mathbb{A} = \langle F; E, \leq, R, S, K, L, M_1, M_2, \ldots, M_7, U\rangle$. The set of variables V has $2^{w_n} - 1$ elements, where the sequence w_n is defined at the beginning of this section. We identify all variables in V with nonempty subsets of w_n, so $V = \{x_A : \emptyset \neq A \subseteq w_n\}$. The binary constraints are as follows:

- Whenever $A_1 \subsetneq A_2$, then the constraint $\rho_{A_1,A_2} = \leq$.
- Whenever $A_1 \cap A_2 = \emptyset$, then the constraint $\rho_{A_1,A_2} = E$.
- Whenever $A_1 \cap A_2 \neq \emptyset$, but A_1 and A_2 are incomparable, then the constraint $\rho_{A_1,A_2} = G$.

Now we need to make sure all possibilities are covered by our ternary constraints. The possible partial orders between three distinct nonempty subsets are

- $A_1 \subsetneq A_2 \subsetneq A_3$, in which case $\rho_{A_1,A_2,A_3} = R_4$.
- $A_1 \subsetneq A_2$ and $A_1 \subsetneq A_3$, while A_2 and A_3 are incomparable, in which case $A_2 \cap A_3 \supseteq A_1 \neq \emptyset$, so $\rho_{A_1,A_2,A_3} = R_5$.
- $A_1 \subsetneq A_3$ and $A_2 \subsetneq A_3$, while A_1 and A_2 are incomparable, in which case either $\rho_{A_1,A_2,A_3} = R_2$ if $A_1 \cap A_2 = \emptyset$, or $\rho_{A_1,A_2,A_3} = R_6$ if $A_1 \cap A_2 \neq \emptyset$.
- $A_1 \subsetneq A_2$ and A_3 is incomparable to either of the A_1 and A_2. Then the three subcases are that $A_3 \cap A_1 = A_3 \cap A_2 = \emptyset$, in which case $\rho_{A_1,A_2,A_3} = R_3$, or that $A_3 \cap A_1 = \emptyset \neq A_3 \cap A_2$, in which case $\rho_{A_1,A_2,A_3} = R_7$, or that that $A_3 \cap A_1 \neq \emptyset \neq A_3 \cap A_2$, in which case $\rho_{A_1,A_2,A_3} = R_{10}$.
- If any pair among A_1, A_2, A_3 are incomparable, then the constraint on those three coordinates is R_1, R_8, R_9, R_{11}, or some permutation of their coordinates, depending on the number of pairs among A_1, A_2, A_3 with nonempty intersection.

Finally, we impose the constraint $\rho_{A_1,A_2,A_3,A_4,A_5,A_6,A_7} = U$ whenever A_1, $\dots, A_7$ are as in the conclusion of Lemma 3.1. From our analysis of the projections of various relations to two-element sets of coordinates and the way the constraints were set up follows that whenever $A \subseteq B$, then the projection of any constraint which contains $\{x_A, x_B\}$ in its scope is $\leq$; whenever $A \cap B = \emptyset$, then the projection of any constraint which contains $\{x_A, x_B\}$ in its scope is E and whenever $A \cap B \neq \emptyset$, but A and B are incomparable, then the projection of any constraint which contains $\{x_A, x_B\}$ in its scope is G. This means that the instance $(V, A, \mathcal{C})$ satisfies the 2-consistency. Moreover, we have exhausted all possibilities of three coordinates, as seen in the above discussion of cases, so there is a ternary constraint on every three-element set of variables, and therefore the instance is $(2, 3)$-minimal. It must have a solution f by Theorem 2.5. By Lemma 3.1, there exist sets $A_1, \dots, A_7$ which are as in the statement of Lemma 3.1, and such that $f(x_{A_i}) = f(x_{A_j})$ for all $1 \leq i < j \leq 7$. So, there must exist some $c \in F$ such that $[c, c, c, c, c, c, c]^T \in U$. Therefore, there must exist a $\mathcal{W}$-term $t(x, y, z, u)$ such that

$$t^{\mathbf{F}^7}\left(\begin{bmatrix} x \\ x \\ x \\ y \\ y \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \\ x \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \\ x \\ x \\ x \end{bmatrix}\right) = \begin{bmatrix} c \\ c \\ c \\ c \\ c \\ c \\ c \end{bmatrix}.$$

This implies that

$$\begin{aligned} \mathbf{F} &\models t(x,x,x,y) \approx t(x,x,y,x) \approx t(x,y,x,x) \approx \\ t(y,x,x,x) &\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x), \text{ and hence} \\ \mathcal{W} &\models t(x,x,x,y) \approx t(x,x,y,x) \approx t(x,y,x,x) \approx \\ t(y,x,x,x) &\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x), \end{aligned}$$

and thus those same identities, together with idempotence of t, hold in $\mathcal{V}$. $\square$

It was proved in [8] that no idempotent strong Mal'cev condition in the language with just one ternary symbol and any number of binary symbols characterizes congruence meet-semidistributivity. So one would need one operation of arity at least 4 to do it, or at least two operations of arity at least 3 each. Therefore, the strong Mal'cev characterization proved above is optimal in the sense of [10].

The condition (SM 4) defined below is the strong Mal'cev condition which was isolated by [8] as the least with respect to the preorder $\preceq$ (i.e. syntactically the weakest) in the class of all strong Mal'cev conditions in the language consisting of two ternary operations, which fail in any nontrivial module, but which are realized in the test varieties which were used in that paper. Of course, those test varieties were all congruence meet-semidistributive and locally finite. We prove that (SM 4) characterizes congruence meet-semidistributivity in locally finite varieties, thus settling the question of number and arities needed for an optimal strong Mal'cev characterization of congruence meet-semidistributivity.

Corollary 3.3. *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ is locally finite and congruence meet-semidistributive, then $\mathcal{V}$ realizes the strong Mal'cev condition* (SM 4) *given by:*

$$\begin{aligned} &p(x,x,x) \approx x \approx q(x,x,x), \\ &p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,x) \text{ and} \\ &q(x,x,y) \approx q(x,y,y). \end{aligned} \tag{SM 4}$$

On the other hand, if $\mathcal{V}$ realizes the strong Mal'cev condition (SM 4)*, then $\mathcal{V}$ is congruence meet-semidistributive.*

Proof. If $\mathcal{V}$ is locally finite and congruence meet-semidistributive, then from Theorem 3.2 follows that $\mathcal{V}$ realizes (SM 3). Then $\mathcal{V}$ realizes (SM 4) using $p = t(x,x,y,z)$ and $q = t(x,y,z,z)$.

On the other hand, let us assume that $p(x,y,z) = \alpha_1 x + \alpha_2 y + \alpha_3 z$, $q(x,y,z) = \beta_1 x + \beta_2 y + \beta_3 z$ satisfy identities (SM 4) in some left $\mathbf{R}$-module $\mathbf{M}$. By evaluating $x = 0$, we get that $\alpha_1 = \alpha_2 = \alpha_3 = \beta_2$ and that $\beta_2 + \beta_3 = \beta_3$, from which follows that $\beta_2 = 0$. Thus $\alpha_1 = \alpha_2 = \alpha_3 = 0$. From idempotence we get that $x = p(x,x,x) = 0x + 0x + 0x = 0$, so $M = \{0\}$. By Theorem 2.6, it follows that $\mathcal{V}$ is congruence meet-semidistributive. $\square$

4. Syntactically stronger and weaker strong Mal'cev characterizations

Besides (SM 3), we can prove another syntactically stronger characterization of locally finite congruence meet-semidistributive varieties than the strong Mal'cev condition (SM 1). This characterization of congruence meet-semidistributivity is not a Mal'cev condition. Instead, it claims that an infinite set of identities in an infinite language must be realized in a variety, or, equivalently, that all members of an infinite sequence of strong Mal'cev conditions are realized in that variety. (Recall that a Mal'cev condition stipulates that at least one of such a sequence holds in the variety.) We will call this kind of condition a *complete Mal'cev condition.*

Proposition 4.1. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff there exists a binary term $t(x,y)$ and for all arities $n \geq 3$ terms $w_n(x_1,\dots,x_n)$ such that*

- *All w_n are weak near-unanimity terms in $\mathcal{V}$ and*
- *For all n, $\mathcal{V} \models w_n(x,x,\dots,x,y) \approx t(x,y)$.* (CM1)

Proof. The strong Mal'cev condition (SM 1) is implied by the complete Mal'cev condition (CM1), so any variety $\mathcal{V}$ which satisfies (CM1) must be congruence meet-semidistributive.

We will prove that for any n_0 there exist $t(x,y)$ such that (CM1) holds for all $3 \leq n \leq n_0$. This will suffice, as the following argument shows: Let every element of $\mathbf{F}_{\mathcal{V}}(x,y)$, represented by the term $t(x,y)$, be assigned a number k which is the least such that for every term operation p of arity k, if p is weak near-unanimity in $\mathcal{V}$, then $\mathcal{V} \not\models p(x,x,\ldots,x,y) \approx t(x,y)$. Our proof will show that for every n_0 there exists t such that t is not assigned any number in the interval $[3,n_0]$. Since $\mathbf{F}_{\mathcal{V}}(x,y)$ is finite, there must exist an element of $\mathbf{F}_{\mathcal{V}}(x,y)$ to which no number is assigned. For every arity $k \geq 3$, therefore, this element is $\mathcal{V}$-equal to the nearly unanimous evaluation of some weak near-unanimity term of arity k.

Now we imitate the proof of Theorem 2.8 given in [13] with the following two modifications: the set of variables is $\{x_1,\ldots,x_n\}$ where $n > (n_0-1)|F_{\mathcal{V}}(x,y)|$ and we impose the appropriate constraints on all subsets of variables with cardinality between 3 and n_0. The proof is now identical to the one in [13]. $\square$

Related to this proposition, we pose an open problem:

Problem 4.2. Can Proposition 4.1 be further strengthened to say that $\mathcal{V} \models t(x,t(x,y)) \approx t(x,y)$, (i. e. so that the weak near-unanimity terms w_n are special in the terminology of [17], Definition 4.6)? What about just realizations of $WNU(k)$ for all $k \geq 3$, without assuming that all derived binary operations are the same (remove the second item from (CM1)) but so that all weak near-unanimity operations are special?

The reason to wonder about this is that the property of being special is often quite useful in the calculations with weak near-unanimity. However, the hitherto known way to prove existence of a special weak near-unanimity term operation in the algebra which has a weak near-unanimity operation is by an iteration which blows up the arity to a factoriel in the exponent. Thus, the above would be quite helpful, if true. Though conceivable in the locally finite case, it is not true in all varieties, as the following example shows:

Example 4.3. Let $\mathbf{A} = \langle \omega; \{s_n^{\mathbf{A}} : n \geq 2\}\rangle$, where the arity of s_n is n and

$$s_n^{\mathbf{A}}(x_1,\ldots,x_n) = \begin{cases} x, \text{ if } x_1 = x_2 = \ldots = x_n = x, \text{ or} \\ \max(x_1,\ldots,x_n)+1, \text{ else.} \end{cases}$$

Proposition 4.4. *The algebra* $\mathbf{A}$ *from Example 4.3 generates a congruence meet-semidistributive variety, but has no special weak near-unanimity terms.*

Proof. Let $\mathcal{V} = \mathcal{V}(\mathbf{A})$. Obviously, all s_n are weak near-unanimity operations with $\mathcal{V} \models s_n(x,x,\ldots,x,y) \approx s_2(x,y)$. Moreover, $\mathcal{V}$ is congruence meet-semidistributive (one can use Theorem 2.8 noting that the direction we need holds in all varieties, or Theorem 3.2 and verify that s_4 satisfies its requirements).

On the other hand, let t be any weak near-unanimity term of $\mathbf{A}$ reduced with respect to idempotence, so any subterm of t of the form $s_i(x_j,x_j,\ldots,x_j)$ is replaced by x_j until none remain. Assume that x_1 occurs in t. Also, let τ be a nearly unanimous evaluation of the variables of t which evaluates all but x_1 as 0, and x_1 as $k \neq 0$. Consider the term tree of t. In the evaluation τ, each node in the term tree is assigned a value equal to the value of the corresponding subterm under τ. Inductively on the depth of a subterm p we can prove that the value assigned to it is equal to k if $p = x_1$, equal to the sum of k and the maximal depth of any occurrence of x_1 if x_1 occurs in p but $p \neq x_1$, and equal to 0 otherwise. By the way, the *depth* of an occurrence of a variable in a term tree is defined as the length (number of covers) of the maximal chain from the root node to the leaf node corresponding to the occurrence, e.g. the depth of x in the term x equal to zero.

Now $t(1, 0, 0, \ldots, 0) = d + 1$, where d is the maximal depth among the occurrences of x_1 in t (since t is weak near-unanimity so $t \neq x_1$). Therefore, $d \geq 1$. On the other hand, $t(t(1, 0, 0, \ldots, 0), 0, 0, \ldots, 0) = 2d + 1 \neq d + 1 = t(1, 0, 0, \ldots, 0)$, so t is not special. □

So, $\mathcal{V}$ is a counterexample to Problem 4.2 which is not locally finite. For this reason, we restricted the scope of Problem 4.2 to locally finite varieties.

Another possible improvement to strong Mal'cev characterizations of locally finite congruence meet-semidistributive varieties we proved so far would be in a reduction of the number of equations needed. We prove first that it is impossible to find such a strong Mal'cev characterization with just idempotence and one more linear equation (thus we may assume that the language has only one operation, too, as applying two distinct operations on the two sides of the equation obviously characterizes nothing):

Theorem 4.5. *Any strong Mal'cev condition in the language with one operation f with arity n which consists of idempotence plus one other linear equation and which is realized in a nontrivial semilattice, can also be realized in a nontrivial module. The module may even be assumed to be finite.*

Proof. Let the strong Mal'cev condition in question be

$$\begin{array}{rcl} f(x, x, \ldots, x) & \approx & x \\ f(y_1, y_2, \ldots, y_n) & \approx & f(z_1, z_2, \ldots, z_n), \end{array} \tag{1}$$

where all y_i and all z_j are in the set $\{x_1, x_2, \ldots, x_m\}$. First of all, we prove that the statement will hold iff it holds under the additional assumption that the identities are *balanced*, i. e. that $\{y_1, y_2, \ldots, y_n\} = \{z_1, z_2, \ldots, z_n\} = \{x_1, x_2, \ldots, x_m\}$.

Assume that $y_t \notin \{z_1, \ldots, z_n\}$. If $J \subseteq \{1, \ldots, n\}$, denote by $\pi_J(y_1, \ldots, y_n)$ the tuple of length $|J|$ consisting of all y_i such that $i \in J$ listed in the increasing order of indices. The Mal'cev condition (1) is realized by a nontrivial semilattice iff the condition

$$\begin{array}{rcl} g(x, x, \ldots, x) & \approx & x \\ g(\pi_J(y_1, y_2, \ldots, y_n)) & \approx & g(\pi_J(z_1, z_2, \ldots, z_n)) \end{array} \tag{2}$$

is realized by a nontrivial semilattice, where $J = \{i : 1 \leq i \leq n \wedge y_i \neq y_t\}$ and the arity of the symbol g is $|J|$. That is because any interpretation of f in a nontrivial semilattice $\mathbf{S}$ must be a meet of variables which are all in J (otherwise the evaluation of y_j as a, and all other variables as b such that $a < b$ would falsify the Mal'cev condition (1)). On the other hand, any algebra in which the Mal'cev condition (2) is realized must realize the original condition (1), by just adding the new dummy variables.

To summarize, if we assume the following implication:

if the condition (2) *is realized in a nontrivial semilattice, then* (2) *is realized in a nontrivial module*,

then we get the implication

if the condition (1) *is realized in a nontrivial semilattice, then* (1) *is realized in a nontrivial module.*

Proof: $\mathbf{S}$ realizes (1) $\Rightarrow$ $\mathbf{S}$ realizes (2) $\Rightarrow$ a module realizes (2) $\Rightarrow$ a module realizes (1). Thus we prune off one after another the variables which occur only on one side. Inductively, we may assume without loss of generality that the equations in condition (1) are balanced.

Next we prove that any balanced condition of the form (1) is realized in the vector space of rational numbers $\mathbf{Q}$ viewed as a space over themselves. If there is any i such that $y_i = z_i$, then just make the interpretation as the ith projection, and this will satisfy the condition (1) in any algebra. Any interpretation $f^{\mathbf{Q}}$ is of the form $f(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \alpha_i u_i$ for some $\alpha_i \in Q$. For any i such that $1 \leq i \leq n$, denote by $I_i = \{j : 1 \leq j \leq n \wedge y_j = x_i\}$ and $J_i = \{j : 1 \leq j \leq n \wedge z_j = x_i\}$.

Claim 1. The condition (1) is realized in an $\mathbf{R}$-module $\mathbf{M}$ iff the system of equations

$$\begin{array}{rclc} \sum_{i=1}^{n} \alpha_i & = & 1 & \text{together with equations} \\ \sum_{j \in I_i} \alpha_j & = & \sum_{j \in J_i} \alpha_j & \text{for each } 1 \leq i \leq m \end{array} \tag{3}$$

has a solution in $\mathbf{R}$ (here α_i are viewed as variables). In one direction by evaluating all x_i as x in (1) from idempotence we get the first equations of (3), while the evaluation of x_i as x and of all other variables x_k as 0 implies the equation $\sum_{j \in I_i} \alpha_j = \sum_{j \in J_i} \alpha_j$. On the other hand, assume that the system (3) has a solution $\langle a_1, \ldots, a_n \rangle$. Interpret $f^{\mathbf{M}}(u_1, \ldots, u_n) = \sum_{i=1}^{n} a_i u_i$. Then

$f^{\mathbf{Q}}(x,\dots,x) = \sum_{i=1}^{n} a_i x = 1x = x$. Moreover,

$$f^{\mathbf{M}}(y_1,\dots,y_n) = \sum_{i=1}^{m}\left(\sum_{j\in I_i}\alpha_j\right)x_i = \sum_{i=1}^{m}\left(\sum_{j\in J_i}\alpha_j\right)x_i = f^{\mathbf{M}}(z_1,\dots,z_n).$$

It remains to show that the system (3) has a solution in $\mathbf{Q}$ no matter which partitions $\{I_i : 1 \le i \le m\}$ and $\{J_i : 1 \le i \le m\}$ the Mal'cev condition (1) imposes. About those partitions, the assumption that $y_i \neq z_i$ for all i reflects as the property that $I_i \cap J_i = \emptyset$ for all i, and this is the only property which we will assume.

We convert the system (3) into the system

$$\begin{array}{rcll} \sum_{i=1}^{n}\alpha_i & = & 1 & \text{and} \\ \sum_{j\in I_i}\alpha_j - \sum_{j\in J_i}\alpha_j & = & 0 & \text{for each } 1 \le i \le m \end{array} \tag{4}$$

Let the matrix M of this system (of dimensions $m+1\times n$) have rank r. Denote by M_1 the matrix obtained from M by deleting the first row. M consists of entries which are 0, 1 or -1, and each column has one 1 in the top row, exactly one more 1 and one -1 and the other entries are zeros. Each row has at least one 1 and, unless it is the first row which consists of all 1s it must also have at least one -1 as a consequence of (1) having balanced identities.

The system (4) will have a solution unless the augmented matrix of the system has rank $r+1$. This will occur iff the column of free coeficients and the first row are used in constructing the minor of order $r+1$ which is regular (since all entries in the column of free coefficients are zeros except for the first one). Computing the determinant of this minor by the last column yields that the augmented matrix of the system has rank $r+1$ iff there is a minor of order r of the matrix M_1 which is regular. Therefore, the system (4) has no solutions in some vector space iff the first row vector of M (consisting of all entries 1) is in the linear span of all other row vectors of M.

Let the row vectors of M_1 be $\mathbf{b}_1,\dots,\mathbf{b}_m$, let $\mathbf{1}$ be the row consisting only of 1s, and let $\mathbf{1} = \sum_{i=1}^{m} q_i\mathbf{b}_i$. If we restrict our attention initially to $\mathbf{Q}$, we may select the value q_k such that $|q_k|$ is maximal. If $q_k > 0$, we know that for some $1 \le j \le m$, $\mathbf{b}_k(j) = -1$. From the way matrix M looks like (and since $\mathbf{b}_i$ are its rows which are not the top one) we know that there is precisely one l such that $\mathbf{b}_l(j) = 1$ and all other rows $\mathbf{b}_i(j) = 0$ whenever $i \neq k$ and $i \neq l$. So we get that

$$1 = \mathbf{1}(j) = \sum_{i=1}^{m} q_i\mathbf{b}_i(j) = q_l - q_k.$$

This implies that $q_l > q_k > 0$ which contradicts the maximality of $|q_k|$. The case when $q_k < 0$ is dealt with analogously, we just need to select a j such that $b_k(j) = 1$ and we will get that $q_l < q_k < 0$, a contradiction again.

So we have proved that the Mal'cev condition (1) is realized in $\mathbf{Q}$ whenever (1) is realized in a nontrivial semilattice. We proceed to prove that it is also realized in $\mathbf{Z}_p$ viewed as a vector space over itself for a suitably selected p. The fact that (1) is realized in $\mathbf{Q}$ implies that the system (4) has a solution $q_1, \ldots, q_n$ in $\mathbf{Q}$. Let k be the positive integer such that all numbers $c_i = kq_i$ are integers. Then the system

$$\begin{array}{rcll} \sum_{i=1}^{n} \alpha_i & = & k & \text{and} \\ \sum_{j \in I_i} \alpha_j - \sum_{j \in J_i} \alpha_j & = & 0 & \text{for each } 1 \leq i \leq m \end{array} \tag{5}$$

has the solution $(c_1, c_2, \ldots, c_n)$ in the ring of integers. Select a prime number p which is relatively prime to k, and for all i, let d_i be the element of Z_p which is congruent to c_i modulo p. Let $l \in Z_p$ be such that lk is congruent to 1 modulo p. Then by multiplying all equations of the system (5) by l in the field $\mathbb{Z}_p$ we get that the system (4), equivalently, the system (3), has the solution $(ld_1, ld_2, \ldots, ld_n)$ in $\mathbf{Z}_p$. So, Claim 1 implies that (1) is realized in $\mathbf{Z}_p$. $\square$

Corollary 4.6. *There exists no idempotent linear strong Mal'cev characterization of locally finite congruence meet-semidistributive varieties in the language with only one operation and one equation other than idempotence.*

Proof. It follows from [7], Theorem 9.10, (2) $\Leftrightarrow$ (5) and Theorem 4.5. $\square$

We are not able to provide any strong Malcev characterization of congruence meet-semidistributive locally finite varieties in the language of a single operation, with idempotence and two more equations. However, a computer search has eliminated all but two conditions with one operation of arity 4 and the two equations having a common term (i. e. of the form $f(\overline{x}) \approx f(\overline{y}) \approx f(\overline{z})$, where $\overline{x}, \overline{y}, \overline{z}$ are some 4-tuples):

$$\begin{array}{rcrcl} t(x,x,x,x) & \approx & x & & \\ t(x,x,y,z) & \approx & t(y,z,y,x) & \approx & t(x,z,z,y) \end{array} \tag{SM 5}$$

$$\begin{array}{rcrcl} t(x,x,x,x) & \approx & x & & \\ t(x,x,y,z) & \approx & t(y,x,z,x) & \approx & t(y,z,x,y) \end{array} \tag{SM 6}$$

Note that both of these conditions are syntactically stronger than (SM 3), i. e. (SM 3) $\preceq$ (SM 5) and (SM 3) $\preceq$ (SM 6). Thus, it would be very desirable that one of them characterizes locally finite congruence meet-semidistributive varieties, as this would constitute the strongest known strong Mal'cev characterization.

In further efforts using a computer search, we found that for all strong Mal'cev conditions in the language of one operation of arity at most 7 and which consist of idempotence and two more equations, all of which use at most three distinct variables, if that strong Mal'cev condition implies congruence meet-semidistributivity, then it either fails in some congruence meet-semidistributive test-algebra, or it is equivalent to (SM 5) or to (SM 6). We leave thus an open problem:

Problem 4.7. Does every locally finite congruence meet-semidistributive variety $\mathcal{V}$ realize the strong Mal'cev condition (SM 5)? What about (SM 6)?

Now we turn to syntactically weaker characterizations. If one is to try to use a computer search to find out whether some locally finite variety is congruence meet-semidistributive, the condition (SM 4) is most useful, as it only requires the computation of the 3-generated free algebra, or the appropriate subalgebra of the third power of the 2-generated free algebra. It is, however, sometimes desirable (e.g. when trying to prove that the locally finite variety is congruence meet-semidistributive, but not by a computer) to find a characterization which is as weak as possible. The conditions (SM 2) and (SM 4) are among the ones with minimal syntactic strength which we are aware of. A sequence $\{\Sigma_k : k \geq 3\}$ of strong Mal'cev characterizations of congruence meet-semidistributivity in locally finite varieties can be defined by changing the arities of the weak near-unanimity terms in (SM 1) to k and $k+1$, respectively. This sequence is of decreasing syntactic strength, as Σ_{k+l} is realized in the variety $\mathrm{Mod}(\Sigma_k)$ by adding l dummy variables to the two weak near-unanimity operations in the definition of Σ_k. However, these are not very useful in practice, syntactically weak as they may be. Once the arity gets large, the problem of finding proofs of existence, whether by computer or by hand, gets technically harder.

So, how weak are these syntactically weak Mal'cev characterizations? We can compare them with other Mal'cev properties which are stronger than congruence meet-semidistributivity. One obvious candidate would be congruence distributivity. We are able to prove that (SM 4) $\preceq CD(4)$ and that (SM 1) $\preceq CD(4)$ (thus also (SM 2) $\preceq CD(4)$, since (SM 2) $\preceq$ (SM 1)):

Proposition 4.8. *Let $\mathcal{V}$ be a variety which realizes $CD(4)$. Then there exist terms $p(x,y,z)$, $q(x,y,z)$ and $w(x,y,z,u)$ such that p and q are a realization of the strong Mal'cev condition* (SM 4) *in $\mathcal{V}$, while p and w are a realization of the strong Mal'cev condition* (SM 1) *in $\mathcal{V}$.*

Proof. Following [17], we introduce the representation of tuples with words, so for instance $a^ib^jca^k$ represents the $(i+j+k+1)$-tuple that has the value a in the first i coordinates, value b in the next j coordinates, value c in the $i+j+1$st coordinate and again a in the final k coordinates.

Let $\mathcal{W}$ be the idempotent reduct of $\mathcal{V}$, as defined at the beginning of the proof of Theorem 3.2. We denote by $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(x,y)$, the $\mathcal{W}$-free algebra freely generated by x and y. We define the following subalgebras of powers of $\mathbf{F}$:

- $G = \mathrm{Sg}^{\mathbf{F}^3}(x^2y, xyx, yx^2)$,
- $H = \mathrm{Sg}^{\mathbf{F}^3}(x^3, yxy, xy^2)$ and
- $K = \mathrm{Sg}^{\mathbf{F}^4}(x^3y, x^2yx, xyx^2, yx^3)$.

We want to prove that there was some $c \in \mathbf{F}$ such that $c^3 \in G \cap H$ and that $c^4 \in K$. This would suffice, since there would exist terms $p(x,y,z), q(x,y,z)$ and $t(x,y,z,u)$ such that $p^{\mathbf{F}^3}(x^2y, xyx, yx^2) = c^3 = q^{\mathbf{F}^3}(x^3, yxy, xy^2)$ and that $w^{\mathbf{F}^4}(x^3y, x^2yx, xyx^2, yx^3) = c^4$. This implies that the desired equations, except for idempotence, hold in $\mathbf{F}$ when we compute the operations $p^{\mathbf{F}^3}$, $q^{\mathbf{F}^3}$ and $w^{\mathbf{F}^4}$ coordinatewise. As $\mathbf{F}$ is the free algebra, and the equalities hold when the terms are applied to the tuples of free generators, this implies that all desired identities hold in $\mathcal{W}$. Idempotence of terms p, q and w in $\mathcal{V}$ (and in $\mathcal{W}$, as well) follows from the definition of $\mathcal{W}$ as the idempotent reduct of $\mathcal{V}$.

Note that G and K are subalgebras of powers of $\mathbf{F}$ which are invariant under all permutations of coordinates (totally symetric subpowers), as explained in Definition 4.2 of [17], so if we prove that, say, $abc \in G$, this will imply that any permutation of the word abc is also in G, and similarly in the case of K.

Now we define three new elements of F: $x_1 := d_1^{\mathbf{F}}(x,x,y) = d_2^{\mathbf{F}}(x,x,y)$, $y_1 := d_2^{\mathbf{F}}(y,x_1,x_1) = d_3^{\mathbf{F}}(y,x_1,x_1)$ and $y_2 := d_2^{\mathbf{F}}(y_1,x_1,x_1) = d_3^{\mathbf{F}}(y_1,x_1,x_1)$. We will prove that $c = y_2$ satisfies the requirements of the first paragraph. First we prove that $y_2^3 \in G$. Note that $\mathbf{G}$ is totally symmetric which we will repeatedly use without mentioning it. Also recall that terms apply coordinatewise like this: $t(abc, def, ghi) = (t(a,d,g), t(b,e,h), t(c,f,i))$.

$$\begin{aligned}
yx_1x &= d_1^{\mathbf{G}}(yxx,\, xxy,\, xyx),\\
yx_1x_1 &= d_1^{\mathbf{G}}(yxx_1,\, xxy,\, xyx_1),\\
y_1x_1x &= d_2^{\mathbf{G}}(yxx,\, x_1xy,\, x_1yx),\\
y_1x_1x_1 &= d_2^{\mathbf{G}}(yxx_1,\, x_1xy,\, x_1yx_1),\\
y_2x_1x_1 &= d_2^{\mathbf{G}}(y_1xx_1,\, x_1xy,\, x_1yx_1),\\
y_1y_2x_1 &= d_3^{\mathbf{G}}(yxx_1,\, x_1xy,\, x_1y_2x_1),\\
y_2y_2x_1 &= d_3^{\mathbf{G}}(y_1xx_1,\, x_1xy,\, x_1y_2x_1),\\
y_2y_2y_2 &= d_3^{\mathbf{G}}(y_1x_1y_2,\, x_1x_1y,\, x_1y_2y_2).
\end{aligned}$$

Next, we prove that $y_2^3 \in H$:

$$\begin{aligned}
x_1xx &= d_1^{\mathbf{H}}(xyy,\, xxx,\, yxy),\\
x_1yy &= d_1^{\mathbf{H}}(xyy,\, xxx,\, yxy),\\
y_1xx &= d_3^{\mathbf{H}}(yxy,\, x_1yy,\, x_1xx),\\
y_1yy &= d_3^{\mathbf{H}}(yxy,\, x_1xx,\, x_1yy).
\end{aligned}$$

Now we let t be a binary term such that $y_2 = t^{\mathbf{F}}(x,y)$. Then

$$\begin{aligned}
x_1y_2y_2 &= t^{\mathbf{H}}(x_1xx,\, x_1yy),\\
y_1y_2y_2 &= t^{\mathbf{H}}(y_1xx,\, y_1yy),\\
y_2y_2y_2 &= d_2^{\mathbf{G}}(y_1y_2y_2,\, x_1y_2y_2,\, x_1y_2y_2).
\end{aligned}$$

Finally, we prove that $y_2^4 \in K$:

$$\begin{aligned}
yx_1x^2 &= d_1^{\mathbf{K}}(yx^3,\, x^3y,\, xyx^2),\\
yx_1^2x &= d_1^{\mathbf{K}}(yxx_1x,\, x^3y,\, xyx_1x),\\
yx_1^3 &= d_1^{\mathbf{K}}(yxx_1^2,\, x^3y,\, xyx_1^2),\\
y_1x_1^2x &= d_2^{\mathbf{K}}(yxx_1x,\, x_1x^2y,\, x_1yx_1x),\\
y_1x_1^3 &= d_2^{\mathbf{K}}(yxx_1^2,\, x_1x^2y,\, x_1yx_1^2),\\
y_2x_1^3 &= d_2^{\mathbf{K}}(y_1xx_1^2,\, x_1xx_1y,\, x_1yx_1^2),\\
y_2^2x_1^2 &= d_3^{\mathbf{K}}(y_1x_1^3,\, x_1^3y,\, x_1y_2x_1^2),\\
y_2^3x_1 &= d_3^{\mathbf{K}}(y_1x_1^3,\, x_1^3y,\, x_1y_2^2x_1),\\
y_2^4 &= d_3^{\mathbf{K}}(y_1x_1^2y_2,\, x_1^3y,\, x_1y_2^3).
\end{aligned}$$

□

The following problem is inspired by Proposition 4.8. It seems to be difficult.

Problem 4.9. (1) Does every congruence distributive variety $\mathcal{V}$ realize the strong Mal'cev condition (SM 4)?

(2) Does every congruence distributive variety $\mathcal{V}$ realize the strong Mal'cev condition (SM 1)?

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Jelena Jovanović

Faculty of Mathematics, Belgrade University, Serbia
e-mail: `jelena.jovanovic55@gmail.com`

Petar Marković

Department of Mathematics and Informatics, University of Novi Sad, Serbia
e-mail: `pera@dmi.uns.ac.rs`

Ralph McKenzie

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
e-mail: `ralph.n.mckenzie@vanderbilt.edu`

Matthew Moore

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
e-mail: `matthew.moore@vanderbilt.edu`